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CYLINDERS IN RATIONAL SURFACES

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ABSTRACT. We answer a question of Ciro Ciliberto about cylinders in rational surfaces which are obtained by blowing up the plane at points in general position.

Let S be a smooth rational surface. A *cylinder* in S is an open subset $U \subset S$ such that

$$U \cong \mathbb{C}^1 \times Z$$

for an affine curve Z . Here, the curve Z is just \mathbb{P}^1 with finitely many missing points.

The surface S contains many cylinders, and it seems hopeless to describe all of them. Instead of doing this, one can consider a similar problem for *polarized* rational surfaces. This problem has a significant application in affine geometry (see [8, 9, 10]).

Fix an ample \mathbb{Q} -divisor A on the surface S .

Definition 1. An A -polar cylinder in S is a Zariski open subset U in S such that

- (C) $U \cong \mathbb{C}^1 \times Z$ for some affine curve Z , i.e., U is a cylinder in S ,
- (P) there is an effective \mathbb{Q} -divisor D on S such that $D \sim_{\mathbb{Q}} A$ and $U = S \setminus \text{Supp}(D)$.

One can always choose an ample divisor A such that S contains an A -polar cylinder. This follows from [8, Proposition 3.13]. On the other hand, we have

Theorem 2 ([10, 2]). *Suppose that S is a smooth del Pezzo surface and $A \in \mathbb{Q}_{>0}[-K_S]$. Then S contains an A -polar cylinder $\iff K_S^2 \geq 4$.*

Theorem 3 ([11, 4]). *Suppose that S is a smooth del Pezzo surface and $A \notin \mathbb{Q}_{>0}[-K_S]$. If $K_S^2 \geq 3$, then S contains an A -polar cylinder.*

The paper [4] also contains a generalization of Theorem 2. To describe it, we put

$$\mu_A = \inf \left\{ \lambda \in \mathbb{Q}_{>0} \mid \text{the } \mathbb{Q}\text{-divisor } K_S + \lambda A \text{ is pseudo-effective} \right\}.$$

It is well-known that $\mu_A \in \mathbb{Q}$ (see, for example, [12, Theorem 2.1] or [13, Theorem 1]). The number μ_A is known as the Fujita invariant of the divisor A (see [7, Definition 3.1]). Let Δ_A be the smallest extremal face of the Mori cone $\overline{\text{NE}}(S)$ that contains $K_S + \mu_A A$. Denote by r_A the dimension of the face Δ_A . Then $r_A + K_S^2 \leq 9$, since S is rational. The number r_A is known as the Fujita rank of the divisor A (see [4, Definition 2.1.1]).

Theorem 4 ([4, Theorem 2.2.3]). *Suppose that S is a smooth del Pezzo surface and*

$$r_A + K_S^2 \leq 3.$$

Then S does not contain A -polar cylinders.

All varieties are assumed to be algebraic, projective and defined over \mathbb{C} .

During the conference *Complex affine geometry, hyperbolicity and complex analysis*, which was held in Grenoble in October 2016, Ciro Ciliberto asked

Question 5. *Suppose that S is a blow up of \mathbb{P}^2 at points in general position and*

$$r_A + K_S^2 \leq 3.$$

Is it true that S does not contain A -polar cylinders?

Ciliberto also suggested to consider Question 5 modulo [5, Conjecture 2.3].

In this paper, we show that the answer to Question 5 is Yes. To be precise, we prove

Theorem 6. *Suppose that S satisfies the following generality condition:*

(★) *the self-intersection of every smooth rational curve in S is at least -1 .*

If $r_A + K_S^2 \leq 3$, then S does not contain A -polar cylinders.

By [6, Proposition 2.4], if S is obtained by blowing up \mathbb{P}^2 at points in general position, then the condition (★) in Theorem 6 is satisfied. Thus, the answer to Question 5 is Yes.

Remark 7. If S is a del Pezzo surface, then the condition (★) in Theorem 6 is satisfied. In fact, if $K_S^2 \geq 1$, then this condition is equivalent to the ampleness of the divisor $-K_S$. This shows that Theorem 6 is a generalization of Theorem 4.

By [9, Corollary 3.2], Theorem 13 implies

Corollary 8. *Suppose that S satisfies the condition (★), and A is a \mathbb{Z} -divisor. Put*

$$V = \operatorname{Spec} \left(\bigoplus_{n \geq 0} H^0(S, \mathcal{O}_S(nA)) \right).$$

If $r_A + K_S^2 \leq 3$, then V does not admit an effective action of the group \mathbb{C}_+ .

The inequality $r_A + K_S^2 \leq 3$ in Theorem 6 is sharp:

Example 9. Suppose that S satisfies the condition (★) in Theorem 6, and $K_S^2 \leq 3$. Then there exists a blow up $f: S \rightarrow \mathbb{P}^2$ of $9 - K_S^2$ different points. Put $k = 4 - K_S^2 \geq 1$. Denote the $9 - K_S^2$ exceptional curves of the blow up f by $E_1, E_2, E_3, E_4, E_5, G_1, \dots, G_k$. Let \mathcal{C} be the unique conic in \mathbb{P}^2 that passes through $f(E_1), f(E_2), f(E_3), f(E_4), f(E_5)$. Let L be a general line in \mathbb{P}^2 tangent to \mathcal{C} , and let \mathcal{P} be the pencil generated by \mathcal{C} and $2L$. Let C_i be the conic in \mathcal{P} that contains $f(G_i)$. Then

$$\mathbb{P}^2 \setminus \left(\mathcal{C} \cup L \cup C_1 \cup \dots \cup C_k \right)$$

is a cylinder. Denote the proper transforms of \mathcal{C} and L on S by $\tilde{\mathcal{C}}$ and \tilde{L} , respectively. Similarly, denote by \tilde{C}_i the proper transform of the conic C_i on the surface S . Then

$$S \setminus \left(\tilde{\mathcal{C}} \cup \tilde{L} \cup E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5 \cup \tilde{C}_1 \cup \dots \cup \tilde{C}_k \cup G_1 \cup \dots \cup G_k \right) \cong \mathbb{P}^2 \setminus \left(\mathcal{C} \cup L \cup C_1 \cup \dots \cup C_k \right).$$

Let ϵ_1, ϵ_2 and x be rational numbers such that $\frac{1}{2} > \epsilon_1 > \frac{\epsilon_2}{2} > 0$ and $1 > x > 1 - \frac{1-2\epsilon_1}{2k}$. Put $A = -K_S + x(G_1 + \cdots + G_k)$. Then A is ample and $r_A = k$, since

$$A \sim_{\mathbb{Q}} \left(1 + \epsilon_1 - \frac{\epsilon_2}{2}\right) \tilde{\mathcal{C}} + \epsilon_2 \tilde{L} + \left(\epsilon_1 - \frac{\epsilon_2}{2}\right) \sum_{i=1}^5 E_i + \frac{1-2\epsilon_1}{2k} \sum_{i=1}^k \tilde{C}_i + \left(x + \frac{1-2\epsilon_1}{2k} - 1\right) \sum_{i=1}^k G_i.$$

Thus, the surface S contains an A -polar cylinder, and $r_A + K_S^2 = 4$.

The inequality $r_A + K_S^2 \geq 4$ does not always imply the existence of A -polar cylinders:

Example 10. Let $f: S \rightarrow \mathbb{P}^2$ be a blow up of 9 points such that $|-K_S|$ is base point free. Then $|-K_S|$ is a pencil. Suppose that all curves in the pencil $|-K_S|$ are irreducible. This easily implies that the surface S satisfies the generality condition (★) in Theorem 6. Suppose, in addition, that all singular curves in the pencil $|-K_S|$ do not have cusps. Let E_1, E_2, E_3 and E_4 be any four f -exceptional curves. Fix $x \in \mathbb{Q}$ such that $0 < x < 1$. Put $A = -K_S + x(E_1 + E_2 + E_3 + E_4)$. Then A is ample. Moreover, we have $r_A = 4$. Furthermore, if $x > \frac{7}{8}$, then it follows from Example 9 that S contains an A -polar cylinder. But S does not contain A -polar cylinders for $x \leq \frac{1}{4}$ by Lemma 12 and Remark 13.

The condition (★) in Theorem 6 cannot be omitted:

Example 11 (cf. [3, Example 4.3]). Let L_1 and L_2 be two distinct lines in \mathbb{P}^2 . Then

$$\mathbb{P}^2 \setminus (L_1 \cup L_2) \cong \mathbb{C}^1 \times \mathbb{C}^*.$$

Let P_1 be a point in $L_1 \setminus L_2$. Let P_2, P_3, P_4, P_5, P_6 and P_7 be general points in $L_2 \setminus L_1$. Let $f: \hat{S} \rightarrow \mathbb{P}^2$ be the blow up of these seven points $P_1, P_2, P_3, P_4, P_5, P_6$ and P_7 . Denote by $F_1, F_2, F_3, F_4, F_5, F_6$ and F_7 the f -exceptional curves such that $f(F_i) = P_i$. Let $g: \tilde{S} \rightarrow \hat{S}$ be the blow up of the point in F_1 contained in the proper transform of L_1 . Denote by G the g -exceptional curve. Let \tilde{F}_1 be the proper transform on \tilde{S} of the curve F_1 . Let $h: \bar{S} \rightarrow \tilde{S}$ be the blow up of the point $\tilde{F}_1 \cap G$. Denote by H the h -exceptional curve. Let $e: \mathcal{S} \rightarrow \bar{S}$ be the blow up of a general point in H . Denote by \mathcal{E} the e -exceptional curve. Denote the proper transforms of the curves $H, G, F_1, F_2, F_3, F_4, F_5, F_6, F_7, L_1$ and L_2 on the surface \mathcal{S} by the symbols $\mathcal{H}, \mathcal{G}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6, \mathcal{F}_7, \mathcal{L}_1$ and \mathcal{L}_2 , respectively. Fix a positive rational number ϵ such that $\epsilon < \frac{1}{3}$. Then

$$-K_{\mathcal{S}} \sim_{\mathbb{Q}} (2 - \epsilon)\mathcal{L}_1 + (1 + \epsilon)\mathcal{L}_2 + (1 - \epsilon)\mathcal{F}_1 + \epsilon \sum_{i=2}^7 \mathcal{F}_i + (2 - 2\epsilon)\mathcal{G} + (2 - 3\epsilon)\mathcal{H} + (1 - 3\epsilon)\mathcal{E}.$$

We also have $\mathcal{S} \setminus (\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{G} \cup \mathcal{H} \cup \mathcal{E}) \cong \mathbb{P}^2 \setminus (L_1 \cup L_2)$. Let $\pi: \mathcal{S} \rightarrow S$ be the contraction of the curves $\mathcal{L}_1, \mathcal{G}$ and \mathcal{H} . Then S is a smooth surface. We have $K_S^2 = 2$, the divisor $-K_S$ is nef and big, and $\pi(\mathcal{E}) \cdot \pi(\mathcal{E}) = \pi(\mathcal{L}_2) \cdot \pi(\mathcal{L}_2) = -2$. In particular, the surface S does not satisfy the generality condition (★) in Theorem 6. Let L_{12} be the line in \mathbb{P}^2 that contains P_1 and P_2 , and let \mathcal{L}_{12} be its proper transform on \mathcal{S} .

Fix a positive rational number x such that $\epsilon > x > 3\epsilon - 1$. Then

$$-K_S + x\mathcal{L}_{12} \sim_{\mathbb{Q}} (2 - \epsilon)\mathcal{L}_1 + (1 + \epsilon)\mathcal{L}_2 + (1 - \epsilon)\mathcal{F}_1 + (\epsilon - x)\mathcal{F}_2 + \\ + \epsilon(\mathcal{F}_3 + \mathcal{F}_4 + \mathcal{F}_5 + \mathcal{F}_6 + \mathcal{F}_7) + (2 + x - 2\epsilon)\mathcal{G} + (2 + x - 3\epsilon)\mathcal{H} + (1 + x - 3\epsilon)\mathcal{E}.$$

Put $A = -K_S + x\pi(\mathcal{L}_{12})$. Then the divisor A is ample and $r_A = 1$, so that $r_A + K_S^2 = 3$. On the other hand, the surface S contains an A -polar cylinder, since

$$A \sim_{\mathbb{Q}} (1 + \epsilon)\pi(\mathcal{L}_2) + (1 - \epsilon)\pi(\mathcal{F}_1) + (\epsilon - x)\pi(\mathcal{F}_2) + \epsilon \sum_{i=3}^7 \pi(\mathcal{F}_i) + (1 + x - 3\epsilon)\pi(\mathcal{E}),$$

and $S \setminus (\pi(\mathcal{L}_2) \cup \pi(\mathcal{F}_1) \cup \pi(\mathcal{F}_2) \cup \pi(\mathcal{F}_3) \cup \pi(\mathcal{F}_4) \cup \pi(\mathcal{F}_5) \cup \pi(\mathcal{F}_6) \cup \pi(\mathcal{F}_7) \cup \pi(\mathcal{E})) \cong \mathbb{C}^1 \times \mathbb{C}^*$.

Recall that S is a smooth rational surface. Let C_1, \dots, C_n be irreducible curves on it. Fix non-negative rational numbers $\lambda_1, \dots, \lambda_n$. Put $D = \lambda_1 C_1 + \dots + \lambda_n C_n$.

In the rest of the paper, we prove Theorem 6. Before doing this, let us prove

Lemma 12. *In the assumptions of Example 10, suppose also that $D \sim_{\mathbb{Q}} A$ and $x \leq \frac{1}{4}$. Then the log pair (S, D) is log canonical.*

Proof. Suppose that (S, D) is not log canonical at some point $P \in S$. Then $\text{mult}_P(D) > 1$. Let \mathcal{C} be the curve in the pencil $| -K_S |$ that contains P . We may assume that $\mathcal{C} = C_1$. Put $\Delta = \lambda_2 C_2 + \dots + \lambda_n C_n$. Then $\lambda_1 < 1$, because

$$C_1 + x(E_1 + E_2 + E_3 + E_4) \sim_{\mathbb{Q}} \lambda_1 C_1 + \Delta.$$

and the log pair $(S, C_1 + x(E_1 + E_2 + E_3 + E_4))$ is log canonical (by assumption).

We have $\lambda_1 > 0$. Indeed, if $\lambda_1 = 0$, then $1 \geq 4x = C_1 \cdot \Delta \geq \text{mult}_P(D) > 1$.

If C_1 is smooth at P , then the inversion of adjunction (see [1, Theorem 7]) gives

$$1 \geq 4x = C_1 \cdot \Delta \geq \left(C_1 \cdot \Delta \right)_P > 1.$$

Thus, the curve C_1 has a simple node at P . Then $P \notin E_1 \cup E_2 \cup E_3 \cup E_4$.

We may assume that one of the curves E_1, E_2, E_3, E_4 is not contained in $\text{Supp}(\Delta)$. Indeed, if this is not the case, then we can swap D with the divisor

$$(1 + \mu)D - \mu \left(C_1 + x(E_1 + E_2 + E_3 + E_4) \right),$$

where μ is the largest positive rational number such that the latter divisor is effective. Without loss of generality, we may assume that $E_4 \not\subset \text{Supp}(\Delta)$. Then

$$1 - x = E_4 \cdot (\lambda_1 C_1 + \Delta) = \lambda_1 + E_4 \cdot \Delta \geq \lambda_1.$$

Put $m = \text{mult}_P(\Delta)$. Then $4x = C_1 \cdot \Delta \geq 2m$, so that $m \leq 2x$.

Let $f: \tilde{S} \rightarrow S$ be the blow up of the point P . Denote by F the f -exceptional curve. Denote by \tilde{C}_1 and $\tilde{\Delta}$ the proper transforms on \tilde{S} of the divisors C_1 and Δ , respectively. Then $(\tilde{S}, \lambda_1 \tilde{C}_1 + \tilde{\Delta} + (2\lambda_1 + m - 1)F)$ is not log canonical at some point $Q \in F$, since

$$K_{\tilde{S}} + \lambda_1 \tilde{C}_1 + \tilde{\Delta} + (2\lambda_1 + m - 1)F \sim_{\mathbb{Q}} f^*(K_S + D).$$

Moreover, we have $2\lambda_1 + m - 1 \leq 1$, since we already proved that $\lambda_1 \leq 1 - x$ and $m \leq 2x$.

If we have $Q \notin \tilde{C}_1$, then $(\tilde{S}, \tilde{\Delta} + F)$ is not log canonical at Q , so that

$$\frac{1}{2} \geq 2x \geq m = F \cdot \tilde{\Delta} \geq (F \cdot \tilde{\Delta})_P > 1$$

by the inversion of adjunction. This shows that $Q \in \tilde{C}_1$.

The curve \tilde{C}_1 is smooth and intersects F transversally at Q . We know that $m \leq 2x \leq 1$. Thus, we can apply [1, Theorem 13] to $(\tilde{S}, \lambda_1 \tilde{C}_1 + \tilde{\Delta} + (2\lambda_1 + m - 1)F)$. This gives

$$4x - 2m = \tilde{\Delta} \cdot \tilde{C}_1 \geq (\tilde{\Delta} \cdot \tilde{C}_1)_Q > 2(1 - (2\lambda_1 + m - 1))$$

or

$$m = \tilde{\Delta} \cdot F \geq (\tilde{\Delta} \cdot F)_Q > 2(1 - \lambda_1).$$

In both cases we obtain a contradiction, since $m \leq 2x$ and $\lambda_1 \leq 1 - x$. \square

Denote by U the complement in the surface S to the union of the curves C_1, \dots, C_n . To prove Theorem 6, we need the following remark originated from [8, Lemma 4.11].

Remark 13 ([4]). Suppose that $U \cong \mathbb{C}^1 \times Z$ for an affine curve Z . Then the exact sequence

$$\bigoplus_{i=1}^n \mathbb{Z}[C_i] \longrightarrow \text{Pic}(S) \longrightarrow \text{Pic}(U)$$

implies that $n \geq \text{rank Pic}(S) = 10 - K_S^2$, because we have $\text{Pic}(U) = 0$, since $\text{Pic}(Z) = 0$. Moreover, the embeddings $Z \hookrightarrow \mathbb{P}^1$ and $\mathbb{C}^1 \hookrightarrow \mathbb{P}^1$ induce a commutative diagram

$$\begin{array}{ccccccc} \mathbb{P}^1 \times \mathbb{P}^1 & \longleftarrow & \mathbb{C}^1 \times \mathbb{P}^1 & \longleftarrow & \mathbb{C}^1 \times Z \cong U & \hookrightarrow & S \\ \downarrow \bar{p}_2 & & \downarrow p_2 & & \downarrow p_Z & & \downarrow \psi \\ \mathbb{P}^1 & \xlongequal{\quad} & \mathbb{P}^1 & \xlongequal{\quad} & \mathbb{P}^1 & \xlongequal{\quad} & \mathbb{P}^1 \end{array}$$

$\swarrow \pi$ $\searrow \phi$
 \mathcal{S}

where p_Z , p_2 and \bar{p}_2 are projections to the second factors, ψ is the map induced by p_Z , the map π is a birational morphism resolving the indeterminacy of ψ , and ϕ is a morphism. Let $\mathcal{E}_1, \dots, \mathcal{E}_m$ be the π -exceptional curves (if π is an isomorphism, we simply put $m = 0$). Let C be the section of the projection \bar{p}_2 , which is the complement of $\mathbb{C}^1 \times \mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Denote by $\mathcal{C}_1, \dots, \mathcal{C}_n$ the proper transforms on \mathcal{S} of the curves C_1, \dots, C_n , respectively. Similarly, denote by the symbol \mathcal{C} the proper transform of the curve C on the surface \mathcal{S} . By construction, a general fiber of the morphism ϕ is \mathbb{P}^1 , and the curve \mathcal{C} is its section. Then the curve \mathcal{C} is either one of the curves $\mathcal{C}_1, \dots, \mathcal{C}_n$ or one of the curves $\mathcal{E}_1, \dots, \mathcal{E}_m$. All the other curves among $\mathcal{C}_1, \dots, \mathcal{C}_n$ and $\mathcal{E}_1, \dots, \mathcal{E}_m$ are mapped by ϕ to points in \mathbb{P}^1 . Without loss of generality, we may assume that either $\mathcal{C} = \mathcal{C}_1$ or $\mathcal{C} = \mathcal{E}_m$. We have

$$K_{\mathcal{S}} + \sum_{i=1}^n \lambda_i \mathcal{C}_i + \sum_{i=1}^m \mu_i \mathcal{E}_i = \pi^* \left(K_S + \sum_{i=1}^n \lambda_i C_i \right)$$

for some rational numbers μ_1, \dots, μ_m . Let \mathcal{F} be a general fiber of ϕ . If $\mathcal{C} = \mathcal{C}_1$, then

$$-2 + \lambda_1 = \left(K_S + \sum_{i=1}^n \lambda_i \mathcal{C}_i + \sum_{i=1}^m \mu_i \mathcal{E}_i \right) \cdot \mathcal{F} = \pi^* \left(K_S + \sum_{i=1}^n \lambda_i C_i \right) \cdot \mathcal{F} = \left(K_S + \sum_{i=1}^n \lambda_i C_i \right) \cdot \pi(\mathcal{F}).$$

Similarly, if $\mathcal{C} = E_m$, then

$$-2 + \mu_m = \left(K_S + \sum_{i=1}^n \lambda_i \mathcal{C}_i + \sum_{i=1}^m \mu_i \mathcal{E}_i \right) \cdot \mathcal{F} = \pi^* \left(K_S + \sum_{i=1}^n \lambda_i C_i \right) \cdot \mathcal{F} = \left(K_S + \sum_{i=1}^n \lambda_i C_i \right) \cdot \pi(\mathcal{F}).$$

In particular, if the divisor $K_S + \sum_{i=1}^n \lambda_i C_i$ is pseudo-effective, then

- if $\mathcal{C} = \mathcal{C}_1$, then $\lambda_1 \geq 2$, so that (S, D) is not log canonical along $C_1 = \pi(\mathcal{C})$,
- if $\mathcal{C} = E_m$, then (S, D) is not log canonical at the point $\pi(\mathcal{C})$.

To prove Theorem 6, we also need

Theorem 14 ([2, Theorem 1.12]). *Suppose that S is a del Pezzo surface, $K_S^2 \leq 3$, and*

$$D \sim_{\mathbb{Q}} -K_S.$$

Let P be a point in S . Suppose that the log pair (S, D) is not log canonical at the point P . Then $|-K_S|$ contains a unique divisor T such that (S, T) is not log canonical at P . Moreover, the support of D contains all the irreducible components of $\text{Supp}(T)$.

Let G_1, \dots, G_r be disjoint smooth rational curves on S such that $G_1^2 = \dots = G_r^2 = -1$. Let $\epsilon_1, \dots, \epsilon_r$ be arbitrary non-negative rational numbers. Suppose that

$$-K_S + \sum_{i=1}^r \epsilon_i G_i \sim_{\mathbb{Q}} D = \sum_{i=1}^n \lambda_i C_i.$$

Lemma 15. *If $C_i \neq G_j$ for all i and j , then (S, D) is log canonical along $G_1 \cup \dots \cup G_r$.*

Proof. Suppose that the log pair (S, D) is not log canonical at some point $P \in G_1 \cup \dots \cup G_r$. Then $\text{mult}_P(D) > 1$. If $P \in G_1$ and $G_1 \not\subset \text{Supp}(D)$, then

$$1 \geq 1 - \epsilon_1 = \left(-K_S + \sum_{i=1}^r \epsilon_i G_i \right) \cdot G_1 = D \cdot G_1 \geq \text{mult}_P(D) > 1.$$

Similarly, we see that $P \notin G_2 \cup \dots \cup G_r$ provided that $C_i \neq G_j$ for all i and j . \square

Let $g: S \rightarrow \bar{S}$ be the contraction of the curves G_1, \dots, G_r . Observe that $K_{\bar{S}}^2 = r + K_S^2$. Put $\bar{C}_1 = g(C_1), \dots, \bar{C}_n = g(C_n)$, and put $\bar{D} = \lambda_1 \bar{C}_1 + \dots + \lambda_n \bar{C}_n$. Then $\bar{D} \sim_{\mathbb{Q}} -K_{\bar{S}}$.

Lemma 16. *Suppose that $r + K_S^2 = 1$, and S satisfies the condition (\star) in Theorem 6. Suppose also that $\lambda_i > 0$ for every i . Then U is not a cylinder.*

Proof. First we observe that the surface \bar{S} also satisfies the condition (\star) in Theorem 6. Then $-K_{\bar{S}}$ is ample by Remark 7. Thus, if $K_{\bar{S}}^2 = 1$, then U is not a cylinder by Theorem 2.

Let us prove the assertion by induction on $K_S^2 \leq 1$. We may assume that $K_S^2 \leq 0$.

Suppose that $C_1 = G_1$. Then there exists a commutative diagram

$$\begin{array}{ccc} & S & \\ f \swarrow & & \searrow g \\ \widehat{S} & \xrightarrow{h} & \overline{S} \end{array}$$

where $f: S \rightarrow \widehat{S}$ is a contraction of the curve $C_1 = G_1$, and h is a birational morphism. Denote by $\widehat{G}_2, \dots, \widehat{G}_r$ the proper transforms on \widehat{S} of the curves G_2, \dots, G_r , respectively. Denote by $\widehat{C}_2, \dots, \widehat{C}_n$ the proper transforms on \widehat{S} of the curves C_2, \dots, C_n , respectively. Then $K_{\widehat{S}}^2 = K_S^2 + 1$ and

$$-K_{\widehat{S}} + \sum_{i=2}^r \epsilon_i \widehat{G}_i \sim_{\mathbb{Q}} \sum_{i=2}^n \lambda_i \widehat{C}_i.$$

By induction $\widehat{S} \setminus (\widehat{C}_2 \cup \dots \cup \widehat{C}_n) \cong U$ is not a cylinder. Thus, we may assume that $C_1 \neq G_1$. Similarly, we may assume that $C_i \neq G_j$ for all possible i and j .

We have $\lambda_i \leq 1$ for each i . Moreover, if $\lambda_i = 1$, then $i = 1$ and $n = 1$, so that $D = C_1$. This follows from the equality $-K_{\overline{S}} \cdot \overline{D} = 1$, because the divisor $-K_{\overline{S}}$ is ample.

Suppose that U is a cylinder. Then $n \geq 10 - K_S^2$ by Remark 13. Then $\lambda_i < 1$ for every i .

Let us use notations of Remark 13. Then $\pi(\mathcal{C})$ is a point. Denote this point by P . By Remark 13 and Lemma 15, (S, D) is not log canonical at P and $P \notin G_1 \cup \dots \cup G_r$.

Put $\overline{P} = g(P)$. Then $(\overline{S}, \overline{D})$ is not log canonical at \overline{P} .

By Theorem 14, there is $\overline{T} \in |-K_{\overline{S}}|$ such that $(\overline{S}, \overline{T})$ is not log canonical at the point \overline{P} . Moreover, Theorem 14 also implies that the curve \overline{T} is one of the curves $\overline{C}_1, \dots, \overline{C}_n$. Therefore, we may assume that $\overline{T} = \overline{C}_1$. Note that the curve \overline{T} is singular at \overline{P} .

Denote by m_1, \dots, m_r the multiplicities of the curve \overline{T} at $g(G_1), \dots, g(G_r)$, respectively. Since the curve \overline{T} is singular at the point \overline{P} by construction, it is smooth away of this point. This implies that either $m_i = 0$ (and $g(G_i) \notin \overline{T}$) or $m_i = 1$ (and $g(G_i) \in \overline{T}$) for all i . Denote by T the proper transform of the curve \overline{T} on the surface S . Then

$$T \sim g^*(\overline{T}) - \sum_{i=1}^r m_i G_i \sim -K_S + \sum_{i=1}^r (1 - m_i) G_i.$$

Let us replace D by a divisor $(1 + \mu)D - \mu T$ for an appropriate rational number $\mu > 0$. The goal is to obtain an effective divisor that do not contain $T = C_1$ in its support. Put

$$D' = \frac{1}{1 - \lambda_1} D - \frac{\lambda_1}{1 - \lambda_1} C_1 = \sum_{i=2}^n \frac{\lambda_i}{1 - \lambda_1} C_i.$$

Then D' is an effective divisor that does not contain $T = C_1$ in its support. But

$$D' \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^r \frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} G_i.$$

Thus, if $\frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} \geq 0$ for every i , then (S, D') is not log canonical at P by Remark 13. In this case, the singularities of the log pair

$$\left(\overline{S}, \sum_{i=2}^n \frac{\lambda_i}{1 - \lambda_1} \overline{C}_i \right)$$

are not log canonical at \overline{P} , since $P \notin G_1 \cup \dots \cup G_r$. The latter is impossible by Theorem 14. Therefore, at least one of the numbers

$$\frac{\epsilon_1 + (m_1 - 1)\lambda_1}{1 - \lambda_1}, \frac{\epsilon_2 + (m_2 - 1)\lambda_1}{1 - \lambda_1}, \dots, \frac{\epsilon_r + (m_r - 1)\lambda_1}{1 - \lambda_1}$$

must be negative. Thus, we may assume that there exists $k \leq r$ such that

$$\frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} < 0$$

for every $i \leq k$, and $\frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} \geq 0$ for every $i > k$ (if any). Then $m_1 = \dots = m_k = 0$. We may also assume that $\epsilon_1 \leq \dots \leq \epsilon_k$. Put

$$D'' = \frac{1}{1 - \epsilon_1} D - \frac{\epsilon_1}{1 - \epsilon_1} C_1 = \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1} C_1 + \sum_{i=2}^n \frac{\lambda_i}{1 - \epsilon_1} C_i.$$

Then D'' is an effective divisor such that

$$D'' \sim_{\mathbb{Q}} -K_S + \sum_{i=2}^r \frac{\epsilon_i - \epsilon_1(1 - m_i)}{1 - \epsilon_1} G_i = \sum_{i=2}^k \frac{\epsilon_i - \epsilon_1}{1 - \epsilon_1} G_i + \sum_{i=k+1}^r \frac{\epsilon_i - \epsilon_1(1 - m_i)}{1 - \epsilon_1} G_i.$$

Let $e: \tilde{S} \rightarrow \overline{S}$ be the blow up of the point $g(G_1)$, and let \tilde{G}_1 be its exceptional curve. Denote by $\tilde{C}_1, \dots, \tilde{C}_n$ the proper transforms on \tilde{S} of the curves C_1, \dots, C_n , respectively. Denote by \tilde{D}'' the proper transform of the divisor D'' on the surface \tilde{S} . Then

$$\tilde{D}'' = \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1} \tilde{C}_1 + \sum_{i=2}^n \frac{\lambda_i}{1 - \epsilon_1} \tilde{C}_i \sim_{\mathbb{Q}} -K_{\tilde{S}}.$$

Since we have $g(G_1) \notin \overline{T}$, the point $g(G_1)$ is not the base point of the pencil $|-K_{\overline{S}}|$. Thus, the pencil $|-K_{\overline{S}}|$ contains a unique irreducible curve \overline{R} that passes through $g(G_1)$. Let \tilde{R} and R be the proper transforms of this curve on the surfaces \tilde{S} and S , respectively. If \overline{R} is singular at $g(G_1)$, then R is a smooth rational curve such that

$$R^2 \leq \tilde{R}^2 = -3.$$

This is impossible by our assumption. Then the curve R is smooth at $g(G_1)$ and $\tilde{R} \sim -K_{\tilde{S}}$. Since $\tilde{R}^2 = 0$ and $\tilde{C}_1 \cdot \tilde{R} = 1$, the divisor \tilde{R} is nef. Then

$$0 = K_{\tilde{S}}^2 = \tilde{D}'' \cdot \tilde{R} = \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1} \tilde{C}_1 \cdot \tilde{R} + \sum_{i=2}^n \frac{\lambda_i}{1 - \epsilon_1} \tilde{C}_i \cdot \tilde{R} \geq \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1} \tilde{C}_1 \cdot \tilde{R} = \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1},$$

so that $\epsilon_1 \geq \lambda_1$. This is a contradiction, since $\epsilon_1 < \lambda_1$. \square

Lemma 17. *Suppose that $r + K_S^2 = 2$, and S satisfies the condition (\star) in Theorem 6. Suppose also that $\lambda_i > 0$ for every i . Then U is not a cylinder.*

Proof. By Remark 7, the surface \bar{S} is a del Pezzo surface. We also have $K_{\bar{S}}^2 = K_S^2 + r = 2$. Therefore, if $K_S^2 = 2$, then $r = 0$ and $S \cong \bar{S}$, so that U is not a cylinder by Theorem 2. Now arguing as in the proof of Lemma 16, we may assume that $C_i \neq G_j$ for all i and j .

By [2, Lemma 3.1] applied to (\bar{S}, \bar{D}) , we have $\lambda_i \leq 1$ for each i .

Suppose that U is a cylinder. Then $n \geq 10 - K_S^2 \geq 9$ by Remark 13.

Let us use notations of Remark 13. Then $\pi(\mathcal{C})$ is a point. Denote this point by P . Then (S, D) is not log canonical at P and $P \notin G_1 \cup \dots \cup G_r$ by Remark 13 and Lemma 2. Put $\bar{P} = g(P)$. Then (\bar{S}, \bar{D}) is not log canonical at \bar{P} .

By Theorem 14, there is a curve $\bar{T} \in |-K_{\bar{S}}|$ such that (\bar{S}, \bar{T}) is not log canonical at \bar{P} , and all irreducible components of the curve \bar{T} are among the irreducible curves $\bar{C}_1, \dots, \bar{C}_n$. In particular, the curve \bar{T} is singular at \bar{P} . It is uniquely determined by this property.

Note that \bar{T} has at most two irreducible component. Thus, we may assume that

- either $\bar{T} = \bar{C}_1$,
- or $\bar{T} = \bar{C}_1 + \bar{C}_2$ and $\lambda_1 \leq \lambda_2$.

If $\bar{T} = \bar{C}_1$, then \bar{T} has a cusp at \bar{P} . If $\bar{T} = \bar{C}_1 + \bar{C}_2$, then \bar{T} has a tacknode at \bar{P} . In both cases, the point \bar{P} is the unique singular point of the curve \bar{T} .

If $\bar{T} = \bar{C}_1$, then $\lambda_1 < 1$, because

$$2 = -K_{\bar{S}} \cdot \bar{D} = \sum_{i=1}^n \lambda_i (-K_{\bar{S}} \cdot \bar{C}_i) = 2\lambda_1 + \sum_{i=2}^n \lambda_i (-K_{\bar{S}} \cdot \bar{C}_i) \geq 2\lambda_1 + \sum_{i=2}^n \lambda_i > 2\lambda_1.$$

Similarly, if $\bar{T} = \bar{C}_1 + \bar{C}_2$, then $\lambda_1 < 1$, because

$$2 = -K_{\bar{S}} \cdot \bar{D} = \lambda_1 + \lambda_2 + \sum_{i=3}^n \lambda_i (-K_{\bar{S}} \cdot \bar{C}_i) > \lambda_1 + \lambda_2 \geq 2\lambda_1.$$

Denote by m_1, \dots, m_r the multiplicities of the curve \bar{T} at $g(G_1), \dots, g(G_r)$, respectively. Then either $m_i = 0$ or $m_i = 1$. Let T be the proper transform of the curve \bar{T} on S . Then

$$T \sim g^*(\bar{T}) - \sum_{i=1}^r m_i G_i \sim -K_S + \sum_{i=1}^r (1 - m_i) G_i.$$

Put $D' = \frac{1}{1-\lambda_1} D - \frac{\lambda_1}{1-\lambda_1} T$ and $\bar{D}' = \frac{1}{1-\lambda_1} \bar{D} - \frac{\lambda_1}{1-\lambda_1} \bar{T}$. If $\bar{T} = \bar{C}_1$, then

$$D' \sim_{\mathbb{Q}} \sum_{i=2}^n \frac{\lambda_i}{1-\lambda_1} C_i.$$

Similarly, if $\bar{T} = \bar{C}_1 + \bar{C}_2$, then

$$D' \sim_{\mathbb{Q}} \frac{\lambda_2 - \lambda_1}{1 - \lambda_1} C_2 + \sum_{i=3}^n \frac{\lambda_i}{1 - \lambda_1} C_i.$$

In both cases D' is an effective divisor that does not contain the curve C_1 in its support. Similarly, the divisor \overline{D}' is effective and does not contain \overline{C}_1 in its support. But

$$D' \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^r \frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} G_i.$$

Thus, if $\frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} \geq 0$ for every i , then (S, D') is not log canonical at P by Remark 13. Then $\overline{D}' \sim_{\mathbb{Q}} -K_{\overline{S}}$ and $(\overline{S}, \overline{D}')$ is not log canonical at \overline{P} , which contradicts Theorem 14. Hence, at least one of the numbers $\frac{\epsilon_1 + (m_1 - 1)\lambda_1}{1 - \lambda_1}, \dots, \frac{\epsilon_r + (m_r - 1)\lambda_1}{1 - \lambda_1}$ is negative.

We may assume that $\frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} < 0 \iff i \leq k$ for some $k \leq r$, and $\epsilon_1 \leq \dots \leq \epsilon_k$. Then $m_i = 0$ and $\epsilon_i < \lambda_1$ for every $i \leq k$.

Put $D'' = \frac{1}{1 - \epsilon_1} D - \frac{\epsilon_1}{1 - \epsilon_1} T$. Then D'' is an effective divisor. Indeed, if $\overline{T} = \overline{C}_1$, then

$$D'' = \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1} C_1 + \sum_{i=2}^n \frac{\lambda_i}{1 - \epsilon_1} C_i.$$

Similarly, if $\overline{T} = \overline{C}_1 + \overline{C}_2$, then

$$D'' = \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1} C_1 + \frac{\lambda_2 - \epsilon_1}{1 - \epsilon_1} C_2 + \sum_{i=3}^n \frac{\lambda_i}{1 - \epsilon_1} C_i.$$

Note that $\text{Supp}(D'') = \text{Supp}(D)$. On the other hand, we have

$$D'' \sim_{\mathbb{Q}} -K_S + \sum_{i=2}^r \frac{\epsilon_i - \epsilon_1(1 - m_i)}{1 - \epsilon_1} G_i.$$

Applying Lemma 16 to D'' , we see that U is not a cylinder. This is a contradiction. \square

Lemma 18. *Suppose that $r + K_S^2 = 3$, and S satisfies the condition (\star) in Theorem 6. Suppose also that $\lambda_i > 0$ for every i . Then U is not a cylinder.*

Proof. By Remark 7, the surface \overline{S} is a cubic surface in \mathbb{P}^3 , because $K_{\overline{S}}^2 = r + K_S^2 = 3$. Moreover, if $K_S^2 = 3$, then $r = 0$ and $S \cong \overline{S}$, so that U is not a cylinder by Theorem 2. Arguing as in the proof of Lemma 16, we may assume that $C_i \neq G_j$ for all possible i and j .

Applying [2, Lemma 4.1] to $(\overline{S}, \overline{D})$, we see that $\lambda_i \leq 1$ for each i .

Suppose that U is a cylinder. Then $n \geq 10 - K_S^2 \geq 8$ by Remark 13.

Let us use notations of Remark 13. Then $\pi(\mathcal{C})$ is a point. Denote this point by P . By Remark 13 and Lemma 2, (S, D) is not log canonical at P and $P \notin G_1 \cup \dots \cup G_r$.

By Theorem 14, there is a curve $\overline{T} \in |-K_{\overline{S}}|$ such that $(\overline{S}, \overline{T})$ is not log canonical at \overline{P} , and all irreducible components of the curve \overline{T} are among the irreducible curves $\overline{C}_1, \dots, \overline{C}_n$. Note that \overline{T} is cut out on $\overline{S} \subset \mathbb{P}^3$ by a hyperplane that is tangent to \overline{S} at the point \overline{P} .

The curve \overline{T} has at most three irreducible component. Thus, we may assume that

- either $\overline{T} = \overline{C}_1$,
- or $\overline{T} = \overline{C}_1 + \overline{C}_2$ and $\lambda_1 \leq \lambda_2$.
- or $\overline{T} = \overline{C}_1 + \overline{C}_2 + \overline{C}_3$ and $\lambda_1 \leq \lambda_2 \leq \lambda_3$.

If $\overline{T} = \overline{C}_1$, then \overline{T} has a cusp at \overline{P} . If $\overline{T} = \overline{C}_1 + \overline{C}_2$, then \overline{T} has a tacknode at \overline{P} . Finally, if $\overline{T} = \overline{C}_1 + \overline{C}_2 + \overline{C}_3$, then the curves \overline{C}_1 , \overline{C}_2 and \overline{C}_3 are lines passing through \overline{P} . In all possible cases, the point \overline{P} is the unique singular point of the curve \overline{T} .

If $\overline{T} = \overline{C}_1$, then $\lambda_1 < 1$, because

$$3 = -K_{\overline{S}} \cdot \overline{D} = \sum_{i=1}^n \lambda_i \left(-K_{\overline{S}} \cdot \overline{C}_i \right) = 3\lambda_1 + \sum_{i=2}^n \lambda_i \left(-K_{\overline{S}} \cdot \overline{C}_i \right) \geq 3\lambda_1 + \sum_{i=2}^n \lambda_i > 3\lambda_1.$$

Similarly, if $\overline{T} = \overline{C}_1 + \overline{C}_2$, then $\lambda_1 < 1$, because

$$3 = \lambda_1 \deg(\overline{C}_1) + \lambda_2 \deg(\overline{C}_2) + \sum_{i=3}^n \lambda_i \left(-K_{\overline{S}} \cdot \overline{C}_i \right) > \lambda_1 \left(\deg(\overline{C}_1) + \deg(\overline{C}_2) \right) = 3\lambda_1.$$

Finally, if $\overline{T} = \overline{C}_1 + \overline{C}_2 + \overline{C}_3$, then we also have $\lambda_1 < 1$, because

$$3 = -K_{\overline{S}} \cdot \overline{D} = \lambda_1 + \lambda_2 + \lambda_3 + \sum_{i=4}^n \lambda_i \left(-K_{\overline{S}} \cdot \overline{C}_i \right) > \lambda_1 + \lambda_2 + \lambda_3 \geq 3\lambda_1.$$

Denote by m_1, \dots, m_r the multiplicities of the curve \overline{T} at $g(G_1), \dots, g(G_r)$, respectively. Then either $m_i = 0$ or $m_i = 1$. Let T be the proper transform of the curve \overline{T} on S . Then

$$T \sim g^*(\overline{T}) - \sum_{i=1}^r m_i G_i \sim -K_S + \sum_{i=1}^r (1 - m_i) G_i.$$

Put $D' = \frac{1}{1-\lambda_1} D - \frac{\lambda_1}{1-\lambda_1} T$ and $\overline{D}' = \frac{1}{1-\lambda_1} \overline{D} - \frac{\lambda_1}{1-\lambda_1} \overline{T}$. If $\overline{T} = \overline{C}_1$, then

$$D' \sim_{\mathbb{Q}} \sum_{i=2}^n \frac{\lambda_i}{1-\lambda_1} C_i.$$

Similarly, if $\overline{T} = \overline{C}_1 + \overline{C}_2$, then

$$D' \sim_{\mathbb{Q}} \frac{\lambda_2 - \lambda_1}{1 - \lambda_1} C_2 + \sum_{i=3}^n \frac{\lambda_i}{1 - \lambda_1} C_i.$$

Finally, if $\overline{T} = \overline{C}_1 + \overline{C}_2 + \overline{C}_3$, then

$$D' \sim_{\mathbb{Q}} \frac{\lambda_2 - \lambda_1}{1 - \lambda_1} C_2 + \frac{\lambda_3 - \lambda_1}{1 - \lambda_1} C_3 + \sum_{i=4}^n \frac{\lambda_i}{1 - \lambda_1} C_i.$$

In all cases D' is an effective divisor that does not contain the curve C_1 in its support. Similarly, the divisor \overline{D}' is effective and does not contain \overline{C}_1 in its support. But

$$D' \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^r \frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} G_i.$$

Thus, if $\frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} \geq 0$ for every i , then (S, D') is not log canonical at P by Remark 13. Then $\overline{D}' \sim_{\mathbb{Q}} -K_{\overline{S}}$ and $(\overline{S}, \overline{D}')$ is not log canonical at \overline{P} , which contradicts Theorem 14. Hence, at least one of the numbers $\frac{\epsilon_1 + (m_1 - 1)\lambda_1}{1 - \lambda_1}, \dots, \frac{\epsilon_r + (m_r - 1)\lambda_1}{1 - \lambda_1}$ is negative.

We may assume that $\frac{\epsilon_i + (m_i - 1)\lambda_1}{1 - \lambda_1} < 0 \iff i \leq k$ for some $k \leq r$, and $\epsilon_1 \leq \dots \leq \epsilon_k$. Then $m_i = 0$ and $\epsilon_i < \lambda_1$ for every $i = 1, \dots, k$.

Put $D'' = \frac{1}{1 - \epsilon_1}D - \frac{\epsilon_1}{1 - \epsilon_1}T$. Then D'' is an effective divisor. Indeed, if $\overline{T} = \overline{C}_1$, then

$$D'' = \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1}C_1 + \sum_{i=2}^n \frac{\lambda_i}{1 - \epsilon_1}C_i.$$

Similarly, if $\overline{T} = \overline{C}_1 + \overline{C}_2$, then

$$D'' = \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1}C_1 + \frac{\lambda_2 - \epsilon_1}{1 - \epsilon_1}C_2 + \sum_{i=3}^n \frac{\lambda_i}{1 - \epsilon_1}C_i.$$

Finally, if $\overline{T} = \overline{C}_1 + \overline{C}_2 + \overline{C}_3$, then

$$D'' = \frac{\lambda_1 - \epsilon_1}{1 - \epsilon_1}C_1 + \frac{\lambda_2 - \epsilon_1}{1 - \epsilon_1}C_2 + \frac{\lambda_3 - \epsilon_1}{1 - \epsilon_1}C_3 + \sum_{i=4}^n \frac{\lambda_i}{1 - \epsilon_1}C_i.$$

In all cases $\text{Supp}(D'') = \text{Supp}(D)$. On the other hand, we have

$$D'' \sim_{\mathbb{Q}} -K_S + \sum_{i=2}^r \frac{\epsilon_i - \epsilon_1(1 - m_i)}{1 - \epsilon_1}G_i.$$

Applying Lemma 17 to D'' , we see that U is not a cylinder. This is a contradiction. \square

Let us show that Theorem 6 follows from Lemmas 16, 17 and 18.

Proof of Theorem 6. Let A be an ample \mathbb{Q} -divisor on S , let μ_A be its Fujita invariant, and let r_A be its the Fujita rank. Suppose that $r_A + K_S^2 \leq 3$. Then

$$\mu_A A \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^{r_A} a_i E_i,$$

where E_1, \dots, E_{r_A} are smooth rational curves on S , and a_1, \dots, a_{r_A} are positive rational numbers. Moreover, there is a birational morphism $f: S \rightarrow \widehat{S}$ such that

- the curves E_1, \dots, E_{r_A} are all f -exceptional curves,
- the surface \widehat{S} is smooth.

This is well-known. See, for example, [12, Theorem 2.1] or [13, Theorem 1].

Suppose, in addition, that the surface S satisfies the condition (\star) in Theorem 6. Then the curves E_1, \dots, E_{r_A} are disjoint and $E_1^2 = E_2^2 = \dots = E_{r_A}^2 = -1$.

To prove Theorem 6, we have to show that S does not contain A -polar cylinders. If $r_A + K_S^2 \geq 1$, then S does not contain A -polar cylinders by Lemmas 16, 17 and 18. Therefore, we may assume that $r_A + K_S^2 \leq 0$. Put $k = 1 - K_S^2 - r_A$.

There is a birational morphism $h: \widehat{S} \rightarrow \widetilde{S}$ such that \widetilde{S} is smooth and $K_{\widetilde{S}}^2 = 1$. Let H_1, \dots, H_k be the proper transforms of the h -exceptional curves on the surface S .

Then the curves H_1, \dots, H_k are smooth rational curves such that $H_1^2 = \dots = H_k^2 = -1$. Moreover, the curves $E_1, \dots, E_{r_A}, H_1, \dots, H_k$ are disjoint. On the other hand, we have

$$\mu_A A \sim_{\mathbb{Q}} -K_S + \sum_{i=1}^{r_A} a_i E_i = -K_S + \sum_{i=1}^{r_A} a_i E_i + 0 \times H_1 + \dots + 0 \times H_k.$$

Therefore, the surface S does not contain A -polar cylinders by Lemma 16. \square

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REFERENCES

- [1] I. Cheltsov, *Del Pezzo surfaces and local inequalities*, Springer Proceedings in Mathematics and Statistics **79** (2014), 83–101.
- [2] I. Cheltsov, J. Park, J. Won, *Affine cones over smooth cubic surfaces*, Journal of the European Mathematical Society **18** (2016), 1537–1564.
- [3] I. Cheltsov, J. Park, J. Won, *Cylinders in singular del Pezzo surfaces*, Compositio Mathematica **152** (2016), 1198–1224.
- [4] I. Cheltsov, J. Park, J. Won, *Cylinders in del Pezzo surfaces*, to appear in International Mathematics Research Notices.
- [5] C. Ciliberto, B. Harbourne, R. Miranda, J. Ro  , *Variations on Nagata’s conjecture*, Clay Mathematics Proceedings **18** (2013), 185–203.
- [6] T. de Fernex, *Negative curves on very general blow-ups of \mathbb{P}^2* , Projective Varieties with Unexpected Properties, 199–207, de Gruyter, Berlin, 2005.
- [7] B. Lehmann, S. Tanimoto, Yu. Tschinkel, *Balanced line bundles on Fano varieties*, to appear in Journal f  r die reine und angewandte Mathematik.
- [8] T. Kishimoto, Yu. Prokhorov, M. Zaidenberg, *Group actions on affine cones*, CRM Proceedings Lecture Notes **54** (2011), 123–163.
- [9] T. Kishimoto, Yu. Prokhorov, M. Zaidenberg, *\mathbb{G}_a -actions on affine cones*, Transformation Groups **18** (2013), 1137–1153.
- [10] T. Kishimoto, Yu. Prokhorov, M. Zaidenberg, *Unipotent group actions on del Pezzo cones*, Algebraic Geometry **1** (2014), 46–56.
- [11] A. Perepechko, *Flexibility of affine cones over del Pezzo surfaces of degree 4 and 5*, Functional Analysis and its Applications **47** (2013), 284289.
- [12] M. Reid, *Surfaces of small degree*, Mathematische Annalen **275** (1986), 71–80.
- [13] F. Sakai, *On polarized normal surfaces*, Manuscripta Mathematica **59** (1987), 109–127.

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